Alternative proof of the example

For a domain $\Omega \subset \mathbb{C}$, $z_0 \in \Omega$, and constants $a \neq 0$ and w_0 with $Re(\bar{a}w_0) > 0$,

$$F = \{f \in H(\Omega) : f(z_0) = w_0, Re(\bar{a}f(z)) > 0, \forall z \in \Omega\}$$

is normal.

Proof.

Method 1: It suffices to show that it is locally bounded. Let K be a compact set of Ω containing z_0. Choose $r > 0$ such that

$$K \subset \cup_{i=1}^{N} B(a_i, r) \subset \cup_{i=1}^{N} \overline{B(a_i, 2r)} \subset \Omega$$

Without loss of generality, we may assume $a_1 = z_0$, and for each j, $a_j \in B(a_i, r)$ for some i. Let $\phi_b : \mathbb{H} \to \mathbb{D}$ to be

$$\phi_b(z) = \frac{z - b}{z + \bar{b}}.$$

Here we denote the right half plane to be $\mathbb{H}$. Thus, the analytic function $g_b(z) = \phi_b(\bar{a}f(z))$ map from Ω to $\mathbb{D}$.

On each $B(a_i, r)$, choose $b = \bar{a}f(a_i)$ in the above equation. We have $g_b : B(a_i, 2r) \to \mathbb{D}$ and $g_b(a_i) = 0$. By Schwarz lemma, we deduce that

$$\left| \frac{\bar{a}(f(z) - f(a_i))}{\bar{a}f(z) + a\overline{f(a_i)}} \right| = |g_b(z)| \leq \frac{|z - a_i|}{2r} \quad \forall z \in B(a_i, 2r)$$

which implies

$$|f(z)| \leq \frac{2r + |z - a_i|}{2r - |z - a_i|} |f(a_i)|.$$

In particular, for all $z \in B(a_i, r)$,

$$|f(z)| \leq 3|f(a_i)|.$$

For each $z \in K$, $z \in B(a_j, r)$ for some j, so we have

$$|f(z)| \leq 3|f(a_j)|.$$

Since there are only finitely many balls covering K, and the number of covering balls is depending on K only, so

$$|f(z)| \leq 3|f(a_j)| \leq 3^N |f(a_1)| = 3^N |w_0|.$$

Method 2: Let $\{f_n\}$ be a sequence in F, consider $g_n = \dfrac{f_n - 1}{f_n + 1} : \Omega \to \mathbb{D}$. We know that $\{g_n\}$ is a bounded analytic function and hence normal. So there exists a subsequence such that g_{n_k} converge to $\phi : \Omega \to \overline{\mathbb{D}}$.

$$\frac{f_{n_k} - 1}{f_{n_k} + 1} \to \phi.$$

If $\phi(a) \neq 1$ for any $a \in \Omega$, we have

$$f_{n_k}(z) \to \frac{\phi(z)+1}{\phi(z)-1}$$

which is holomorphic. If $\phi(a) = 1$ for some $a \in \Omega$, by maximum principle, we conclude that

$$\phi(z) \equiv 1 \Rightarrow \phi(z_0) = 1 = \frac{w_0 - 1}{w_0 + 1}.$$

Contracidtion arised. So $\{f_n\}$ is normal. That is F is normal.

□